Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.

Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.

April 2013 **MTMD03**

Answer Book
Any bilingual English language dictionary permitted

UNIVERSITY OF READING

Monte-Carlo Methods and Particle Filters (MTMD03)

Two hours

Answer **ANY TWO** questions

The marks for the individual components of each question are given in [ ] brackets. The total mark for the paper is 100.

1. a) Show how to derive samples from the Generalised extreme value (GEV) distribution density using the transformation method, also called probability integral transform method, starting from samples from the uniform distribution U(0,1].
The GEV density is given by:

$$f(x) = \frac{1}{\sigma}\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\exp\left[-\exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]\right]$$

Hint: use integration variable $y = \exp\left[-\left(\frac{x-\mu}{\sigma}\right)\right]$

[10 marks]

b) Consider a Markov chain on an *n* dimensional system with transition probability matrix

$$P = \begin{pmatrix} q & p & 0 & \cdots & \cdots & 0 \\ q & 0 & p & \ddots & & \vdots \\ 0 & q & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & p & 0 \\ \vdots & & \ddots & q & 0 & p \\ 0 & \cdots & \cdots & 0 & q & p \end{pmatrix}$$

so zeros on the main diagonal, apart fro the first and last element, *p* on the first upper diagonal, *q* on the first lower diagonal, and the rest zeros. What condition is needed on *p* and *q* to make this a proper stochastic matrix?

[4 marks]

c) What is the probability to move from x_1 to x_2 in one step? And from x_5 to x_1 (assuming n>5)?

[6 marks]

d) What is the probability to end up in x_k from any other position in one step, for *k* not equal to 1 or *n*? Use $p+q=1$.

[5 marks]

e) Show that the Markov chain is ergodic.

[5 marks]

f) Find the invariant distribution *p(x)*.

[10 marks]

g) Show that if $p=q$ all states are equally likely to be visited, but if $p<q$ the states with small k are more likely than those with large k.

[10 marks]

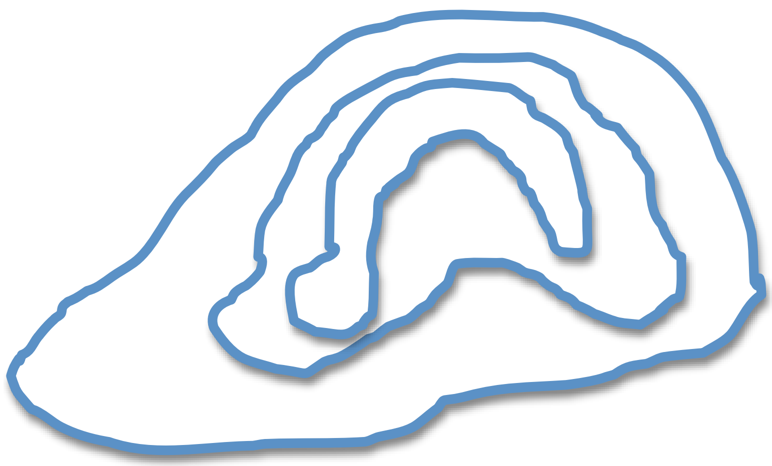
2. a) Explain what Markov Chain Monte-Carlo methods are, and why they can be useful for the data-assimilation problem.

[10 marks]

b) Why are successive samples from a Markov Chain not independent? How can we obtain independent samples?

[10 marks]

c) Consider the contour plot of the posterior probability density depicted in the figure. Why would standard Metropolis-Hastings with random-walk sampling not be efficient?



[10 marks]

d) Explain in words why Hybrid Monte-Carlo is more efficient than standard Metropolis-Hastings.

[10 marks]

e) Why does one need multiple steps of the Hamiltonian system to generate one sample?

[10 marks]

3. Suppose we want to calculate the integral

$$I = \int_{-10}^{10} f(x)dx$$

in which $f(x) = \exp\left(-x^2\right)$

a) Explain why dividing the domain up in fixed size intervals Dx and do the integration as a summation of *f(x)* over these intervals will not be very efficient, that is a lot of function evaluations are needed to make the estimate accurate.

[10 marks]

b) Show how the integration in a) can be interpreted as importance sampling with a uniform density as proposal.

[5 marks]

c) Show how to use the proposal density

$$q(x) = \frac{1}{1+x^2}$$

to generate the samples, and that this will be much more efficient. Use the variance of the Monte-Carlo estimate to prove the latter.

[10 marks]

d) Explain why standard Particle Filters are not efficient when the likelihood is much more peaked than the prior, and explain how a proposal density can be used to make Particle Filters more efficient.

[10 marks]

Assume a numerical model of the form

$$x^n = f(x^{n-1}) + \beta^n$$

in which *n* is the time index, *f(..)* is the deterministic model, and b^n is the random forcing, drawn from a Gaussian distribution with mean zero and covariance *Q*.

e) Give an expression for

$p\left(x^n \mid x^{n-1}\right)$

for this model.

[5 marks]

f) Give an example of a proposal density that uses observations at time *m*>*n*, and give an expression for the posterior weights when this proposal density is used.

[10 marks]

[End of Question paper]