**Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.**

**Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.**

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**April 2013** **MTMD02**

**Answer Book**
**Data Sheet**
**Lecture notes permitted**
**Any bilingual English language dictionary permitted**
**Only Casio-fx83 calculators are permitted**

**UNIVERSITY OF READING**

**Operational Data Assimilation Techniques (MTMD02)**

Two hours

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Answer **ANY TWO** questions

The marks for the individual components of each question are given in [ ] brackets. The total mark for the paper is 100.

1. The evolving temperature of an air mass is to be estimated by data assimilation. By making the assumption that the air mass has a uniform temperature, the temperature may be described by a single variable, $T(t)$, at each time $t$.

   A model for how the temperature changes in time is Newtonian cooling, which has the following form:

$$T(t) = T_{\text{env}} + (T(0) - T_{\text{env}})\exp(-\alpha t),$$

   where $T_{\text{env}}$ is the environmental temperature and $\alpha$ is the cooling rate (both are positive constants and are known perfectly). A temperature observation is made at each of the times $t_1$ and $t_2$ ($y(t_1)$ and $y(t_2)$ respectively), where $t_1, t_2 \geq 0$. $T(0)$ is to be estimated by strong constraint 4D-VAR using the observations and a background state, $T_{\text{B}}(0)$.

   a) What is meant by the terms "4D-VAR" and "strong constraint"? [3 marks]

   b) In the context of variational data assimilation, what is a control variable? [2 marks]

   c) Write down the observation operators for observations 1 and 2 that may be used in strong constraint 4D-VAR with $T(0)$ as the control variable. [2 marks]

   d) A 4D-VAR cost function is given as

$$J\left(T(0)\right) = \frac{1}{2}\left[\frac{\left(T(0) - T_B(0)\right)^2}{\sigma_\alpha^2} + \frac{\left(y(t_1) - y^m(t_1)\right)^2}{\sigma_\beta^2} + \frac{\left(y(t_2) - y^m(t_2)\right)^2}{\sigma_\gamma^2}\right],$$

   Identify the meaning of each symbol $\sigma_\alpha$, $\sigma_\beta$, $\sigma_\gamma$, and $y^m(t_n)$ in the context of the previous parts of the question. [8 marks]

e) Write down the derivative of the cost function with respect to $T(0)$ either by differentiating it by hand or by using a standard formula for the gradient. [5 marks]

f) Find the analysis state that minimizes the cost function. [10 marks]

g) Show that in the limit that observation 1 is much more accurate than both the background state and observation 2, the analysis is
$T(0) = (y(t_1) - T_{\text{env}})\exp(\alpha t_1) + T_{\text{env}}$. [4 marks]

h) Comment on why you think the impact that this observation has on the analysis goes as $\exp(\alpha t_1)$ and state why this poses a potential difficulty for the assimilation of observations made at distant times. [8 marks]

i) A 3D-VAR system is to be developed from the 4D-VAR system used above. This system is identical in every way except that the observation operator does not contain a part that propagates the state in time. From your answer to part (f) write down the analysis that you would get if this 3D-VAR system were used instead of 4D-VAR and suggest a simple mathematical condition that would have to be satisfied for 3D-VAR to be a good approximation to 4D-VAR. [8 marks]

2. a) The 3D-VAR cost function may be written as the following two terms

$$J(\mathbf{x}) = \frac{1}{2}(\mathbf{x}-\mathbf{x}_{\mathrm{B}})^{\mathrm{T}}\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_{\mathrm{B}}) + \frac{1}{2}(\mathbf{y}-\mathbf{h}(\mathbf{x}))^{\mathrm{T}}\mathbf{R}^{-1}(\mathbf{y}-\mathbf{h}(\mathbf{x})),$$

$$= \qquad \text{term 1} \qquad + \qquad \text{term 2}$$

where there are $n$ unknowns in $\mathbf{x}$ and $m$ observations.

i. By assuming the usual meanings, explain the meaning of each symbol: $J$, $\mathbf{x}$, $\mathbf{x}_{\mathrm{B}}$, $\mathbf{y}$, $\mathbf{h}$, $\mathbf{B}$ and $\mathbf{R}$. Include whether each represents a scalar, vector, matrix or function. For vectors and matrices include their dimensions, and for functions include the dimensions of their inputs and outputs. [14 marks]

ii. Explain the purpose of each term in this cost function. [2 marks]

Under a certain condition, the shape of $J$ as a function of $\mathbf{x}$ is quadratic.

iii. What is this condition? [1 mark]

iv. Mention the main problem that may arise when this condition is not met. [1 mark]

$\mathbf{h}$ may be expanded in the following way:

$$\mathbf{h}(\mathbf{x}_{\mathrm{B}} + \delta\mathbf{x}) \approx \mathbf{h}(\mathbf{x}_{\mathrm{B}}) + \mathbf{H}\delta\mathbf{x},$$

where $\delta\mathbf{x}$ is a perturbation to $\mathbf{x}$ and $\mathbf{H}$ is the Jacobian of $\mathbf{h}$ evaluated at $\mathbf{x}_{\mathrm{B}}$.

v. By writing $\mathbf{x} = \mathbf{x}_{\mathrm{B}} + \delta\mathbf{x}$ derive an incremental form of the 3D-VAR cost function. [5 marks]

vi. Explain briefly how the incremental form helps solve the problem mentioned in part (iv) for non-linear **h** operators.

[2 marks]

b) Consider a time interval and assume that a forecast of the state of a system (e.g., the atmosphere) and a set of observations valid for the same time interval are available. Consider also an estimate of the state that combines the information from the forecast and the available observations.

i. Discuss how observations are used to determine the estimate of the state when the estimate is derived using a smoother and a filter. [5 marks]

ii. In the case when the forecast model is linearly related to the state, is there a time within the interval when the estimate from the smoother coincides with that from a filter? Justify your answer. [5 marks]

iii. When the evolution of the state is described by a nonlinear model, name at least one method to determine an estimate of the state with a smoother and one with a filter. Briefly discuss the characteristics of the two methods. [7 marks]

iv. Discuss the differences between the minimum variance estimate and the conditional mode estimate. Provide a sufficient condition for the two estimates to coincide.

[8 marks]

3. Consider a discrete-in-time scalar process x(t) that is constant in time, described as $x_{k+1} = x_k$ , where $x_k \equiv x(t_k)$.

a) Write a differential equation that represents a continuous-in-time scalar process that is constant in time. Derive the expression of the discrete-in-time scalar process given above using the Euler method. [5 marks]

Now assume that at all discrete times, except for time $t_0$, a direct measurement of $x_k$ is available and is described by $y_k = x_k + \varepsilon^0$, where $\varepsilon^0$ is a Gaussian measurement noise, which is unbiased, with variance r and is temporally uncorrelated and uncorrelated with the initial state $x_0$.

b) Write the expression for $p_{k+1}^f$, the variance of $x_{k+1}$ at time $t_{k+1}$, as a function of the variance $p_{k+1}^a$ (assumed known) of the best estimate of $x_k$ [5 marks]

c) Write the Kalman filter expression for the variance of the best estimate of x at time $t = t_1$ after the measurement $y_1$ is taken. [8 marks]

d) By using the Kalman filter update equation, write an expression for $x_1^a$, the best estimate of x(t) at time $t = t_1$ after the measurement $y_1$ is taken. Then assume that $x_0^a$, the best estimate at t=0, is given by $x_0^a = 1$ with variance $p_0^a = 2$ and that $y_1 = 1.5$ and r = 1. Calculate $x_1^a$. [8 marks]

e) Assume the initial variance of $x_0^a$ is $p_0^a = 2$ and consider a constant r = 1, as in d). Determine the time index $k_c$ after which the multiplicative weight to the observation value in the Kalman filter update equation becomes smaller than 0.4.

[10 marks]

Consider now the case when the evolution of the state $x_t$ in a) is also affected by Gaussian noise $w_t$ with mean zero and constant variance q.

f) Write a discrete-time equation that describes the evolution of $x_k = x(t_k)$ [5 marks]

g) By using the Kalman filter update equation, write an expression for $x_1^a$, the best estimate of $x_t$ at time $t = t_1$ after the measurement $y_1$ is taken. Then assume that $x_0^a$, the best estimate at t=0, is given by $x_0^a = 1$ with variance $p_0^a = 2$ and that $y_1 = 1.5$, r = 1 and q = 1. Calculate $x_1^a$. [9 marks]

[End of Question paper]