Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.

Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.

April 2012

Answer Book
Data Sheet
Lecture notes permitted
Any bilingual English language dictionary permitted
Only Casio-fx83 calculators are permitted

Final Examination for MSc

Course in Data Assimilation and Inverse Modelling in Geosciences

MTMD02

Operational Data Assimilation Techniques

Two hours

Answer **ANY TWO** questions

The marks for the individual components of each question are given in [ ] brackets. The total mark for the paper is 100.

1. Temperature and pressure fields in a vertical column of the atmosphere are represented by the state vector $\mathbf{x}$ which has $n/2$ grid points representing temperature and $n/2$ grid points representing pressure. The grid has levels at heights z_i where i is the level index. An analysis ($\mathbf{x}^{\mathrm{a}}$) is required from observations ($\mathbf{y}$) using a multivariate data assimilation system.

(a) In practice a background state ($\mathbf{x}^{\mathrm{b}}$) is needed. Explain what the background state is, why it is needed and how it is determined for use in data assimilation.

[3 marks]

(b) $\mathbf{x}$, $\mathbf{x}^{\mathrm{a}}$ and $\mathbf{x}^{\mathrm{b}}$ are to be represented by the augmented vectors

$$\mathbf{x} = \begin{pmatrix} \mathbf{x}_T \\ \mathbf{x}_p \end{pmatrix}, \quad \mathbf{x}^{\mathrm{a}} = \begin{pmatrix} \mathbf{x}_T^{\mathrm{a}} \\ \mathbf{x}_p^{\mathrm{a}} \end{pmatrix}, \quad \mathbf{x}^{\mathrm{b}} = \begin{pmatrix} \mathbf{x}_T^{\mathrm{b}} \\ \mathbf{x}_p^{\mathrm{b}} \end{pmatrix},$$

where $\mathbf{x}_T$ and $\mathbf{x}_p$ are representations of the vertical profiles of temperature and pressure respectively. The background error covariance matrix is $\mathbf{B}$. Elements of $\mathbf{B}$ between background errors of pressure at levels i and j are $\mathbf{B}_{pp}(z_i, z_j)$ and have the following form

$$\mathbf{B}_{pp}(z_i, z_j) = \sigma_p^2 \mu(z_i, z_j), \text{ where } \mu(z_i, z_j) = \exp - \frac{(z_i - z_j)^2}{2L^2},$$

where σ_p is the (constant) pressure error standard deviation and L is the vertical correlation length.

(i) Explain the meaning of $\mathbf{B}_{pp}(z_i, z_j)$ and the role of σ_p and L in data assimilation.

[4 marks]

(ii) Just as $\mathbf{x}$ has the augmented form given above, write down the corresponding augmented form of $\mathbf{B}$ in terms of $\mathbf{B}_{TT}$ (the covariance matrix for temperature background errors) and $\mathbf{B}_{Tp}$ (the covariance matrix between temperature and pressure background errors) and $\mathbf{B}_{pp}$.

[4 marks]

First an analysis is required of pressure only from a single pressure observation. The observation (y_p) is made at height z_k, at a time very close to the validity time of the background, and with instrument error standard deviation s_p. The operator $\mathbf{H}$ acts on a state vector to give the model's version of the observation.

(iii) Given that y_p is a direct observation at height z_k, write down elements of $\mathbf{H}$.

[6 marks]

(iv) Using this $\mathbf{H}$, find $\mathbf{BH}^{\mathrm{T}}$ and $\mathbf{HBH}^{\mathrm{T}}$.

[8 marks]

(v) The optimal interpolation formula is

$$\mathbf{x}^{\mathrm{a}} = \mathbf{x}^{\mathrm{b}} + \mathbf{BH}^{\mathrm{T}}(\mathbf{R} + \mathbf{HBH}^{\mathrm{T}})^{-1}\left(\mathbf{y} - \mathbf{Hx}^{\mathrm{b}}\right),$$

Where $\mathbf{y}$ is a general vector of observations and $\mathbf{R}$ is the observation error covariance matrix. Use this formula with your results from part (iv) to give the <u>pressure only</u> analysis as a function of height. $\mathbf{B}$ has the form given above.

[5 marks]

(c) Background errors in pressure $\delta p^{\mathrm{b}}(z)$, and temperature, $\delta T^{\mathrm{b}}(z)$ are assumed to be close to hydrostatic balance, which is to be modelled by

$$\delta T^{\mathrm{b}}(z) = \varepsilon\left(\lambda\frac{\partial}{\partial z} - 1\right)\delta p^{\mathrm{b}}(z),$$

where z is height and ε and λ are constants. The pressure-pressure errors are represented by $\langle \delta p^{\mathrm{b}}(z_i)\delta p^{\mathrm{b}}(z_j)\rangle = \mathbf{B}_{pp}(z_i, z_j) = \sigma_p^2\mu(z_i, z_j)$, where μ is given above and $\langle\quad\rangle$ means 'expectation'. Use this definition together with the hydrostatic relation to derive expressions for elements of the pressure-temperature, temperature-pressure, and temperature-temperature background errors between positions z_i and z_j as follows

$$\langle \delta p^{\mathrm{b}}(z_i)\delta T^{\mathrm{b}}(z_j)\rangle = \mathbf{B}_{pT}(z_i, z_j),$$
$$\langle \delta T^{\mathrm{b}}(z_i)\delta p^{\mathrm{b}}(z_j)\rangle = \mathbf{B}_{Tp}(z_i, z_j), \text{ and}$$
$$\langle \delta T^{\mathrm{b}}(z_i)\delta T^{\mathrm{b}}(z_j)\rangle = \mathbf{B}_{TT}(z_i, z_j).$$

The following derivatives are provided for convenience

$$\frac{\partial\mu(z_i, z_j)}{\partial z_i} = -\mu(z_i, z_j)\frac{z_i - z_j}{L^2},$$
$$\frac{\partial\mu(z_i, z_j)}{\partial z_j} = \mu(z_i, z_j)\frac{z_i - z_j}{L^2}, \text{ and}$$
$$\frac{\partial^2\mu(z_i, z_j)}{\partial z_i\partial z_j} = \frac{\mu(z_i, z_j)}{L^2} - \mu(z_i, z_j)\frac{(z_i - z_j)^2}{L^4}.$$

[18 marks]

(e) Give two sources of error that have been ignored in the above data assimilation system. [2 marks]

2.

(a) A 4-D VAR cost function has the following incremental form

$$J[\delta\mathbf{x}] = \frac{1}{2}\delta\mathbf{x}^{\mathrm{T}}\mathbf{B}^{-1}\delta\mathbf{x} + \frac{1}{2}\sum_{t=0}^{T}(\delta\mathbf{y}_t - \mathbf{H}_t\mathbf{M}_{t\leftarrow 0}\delta\mathbf{x})^{\mathrm{T}}\,\mathbf{R}_{\mathrm{t}}^{-1}(\delta\mathbf{y}_t - \mathbf{H}_t\mathbf{M}_{t\leftarrow 0}\delta\mathbf{x})$$

where $\delta\mathbf{x} = \mathbf{x} - \mathbf{x}^{\mathrm{b}}$, $\mathbf{x}^{\mathrm{b}}$ is the background state, $\delta\mathbf{y}_t = \mathbf{y}_t - H_t(M_{t\leftarrow 0}(\mathbf{x}^{\mathrm{b}}))$, $\mathbf{y}_t$ is the observation vector at time t, h_t and $\mathbf{H}_t$ is the observation operator and its Jacobian respectively, $M_{t\leftarrow 0}$ and $\mathbf{M}_{t\leftarrow 0}$ is the forecast model and its Jacobian respectively, $\mathbf{B}$ is the background error covariance matrix, and $\mathbf{R}_t$ is the observation error covariance matrix at time t.

(i) Write down the Hessian matrix of $J[\delta\mathbf{x}]$ with respect to $\delta\mathbf{x}$.

[2 marks]

Consider a change of variable from $\delta\mathbf{x}$ to $\delta\boldsymbol{\chi}$, where $\delta\mathbf{x} = \mathbf{U}\delta\boldsymbol{\chi}$, $\mathbf{U}$ being a full-rank linear operator.

(ii) Write down the cost function written in terms of $\delta\boldsymbol{\chi}$, $J[\delta\boldsymbol{\chi}]$ instead of $\delta\mathbf{x}$ without assuming any particular form of $\mathbf{U}$.

[3 marks]

(iii) Write down the Hessian matrix of $J[\delta\boldsymbol{\chi}]$ with respect to $\delta\boldsymbol{\chi}$.

[3 marks]

A form of $\mathbf{U}$ is now taken to be a Cholesky decomposition of $\mathbf{B}$, meaning that $\mathbf{U}$ is designed to satisfy $\mathbf{B} = \mathbf{U}\mathbf{U}^{\mathrm{T}}$.

(iv) Write down the Hessian matrix of $J[\delta\boldsymbol{\chi}]$ with respect to $\delta\boldsymbol{\chi}$ when $\mathbf{U}$ has this form.

[2 marks]

(v) With reference to your answers to parts (i) and (iv), comment on why in practice the cost function is minimized with respect to $\delta\boldsymbol{\chi}$ (with $\mathbf{U}$ satisfying $\mathbf{B} = \mathbf{U}\mathbf{U}^{\mathrm{T}}$) rather than with respect to $\delta\mathbf{x}$.

[3 marks]

Let $\delta\mathbf{x}$ represent a 1-D domain of a single scalar variable with n components, $\delta x(r_p)$ where $r_p = p\Delta r$ is the position of the pth grid point, which are separated by distance Δr. Let $\delta\boldsymbol{\chi}$ represent a vector of scaled Fourier components of $\delta\mathbf{x}$, each component being $\delta\chi(k_q)$. Let $\mathbf{U}$ represent the following Fourier series operator with scaling, $\lambda^{\frac{1}{2}}(k_q)$

$$\delta x(r_p) = \frac{1}{\sqrt{n}}\sum_{q=1}^{n} \delta\chi(k_q)\lambda^{\frac{1}{2}}(k_q)\exp(ik_q r_p),$$

where $i = \sqrt{-1}$. The complex adjoint of this operator is

$$\delta\chi(k_q) = \lambda^{\frac{1}{2}}(k_q)\frac{1}{\sqrt{n}}\sum_{p=1}^{n} \delta x(r_p)\exp(-ik_q r_p).$$

(vi) Compute the element of the $\mathbf{B}$ -matrix between locations $r_{p\prime}$ and r_p, $\mathbf{B}(r_{p\prime}, r_p)$ implied by the above transforms and show that this implied $\mathbf{B}$ -matrix is homogeneous. [12 marks]

(b) Consider the linear dynamic system defined by the linear first-order system of differential equations given by

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t)$$

where $\mathbf{A} \in \Re^{n\times n}$ and $\mathbf{x} \in \Re^n$.

(i) Write a formal expression involving $\mathbf{A}$ that determines $\mathbf{x}$(t) from a set of initial conditions $\mathbf{x}(0)$. [5 marks]

(ii) Discuss the main properties of the function that determines $\mathbf{x}$(t) from $\mathbf{x}(0)$. [8 marks]

(iii) Consider $\mathbf{A} \in \Re^{2\times 2}$ and $\mathbf{x} \in \Re^2$. Now assume that if

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \text{ then } \mathbf{x}(t) = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix}$$

and if $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then $\mathbf{x}(t) = \begin{pmatrix} e^{t} \\ e^{t} \end{pmatrix}$.

Determine the transition matrix $\Phi(t,0)$ for the system and verify that $\Phi(0,0) = \mathbf{I}$. [12 marks]

3. Let us assume that the evolution of the state x_{i+1} and the observation y_{i+1} at discrete time t_{i+1} are given by the scalar relations

$$x_{i+1} = ax_i + \eta_i$$

$$y_{i+1} = hx_{i+1} + \varepsilon_{i+1}$$

where $a \neq 0$ and $h \neq 0$ and where η_i and ε_{i+1} are white Gaussian noise processes with zero mean and variance q and r, respectively. It is also assumed that x_0 at time t_0 has mean m_0 and variance p_0 and that x_0, η_i and ε_{i+1} are mutually uncorrelated.

a) Assume that x_i^a is the best estimate of x_i at time t_i, defined as the expectation of x_i, conditioned on all available observations up to time t_i. Determine the expression for the best estimate of the state x_{i+1}^f and its uncertainty (i.e., one error standard deviation) at time t_{i+1} in the case when no measurements at time t_{i+1} are available.

[10 marks]

b) Choose $a = 2$, $m_0 = p_0 = q = 1$ and produce a sketch of the trajectory of x_{i+1}^f (x_{i+1}^f versus t_i) and its uncertainty, between $t=t_0$ and $t=t_2$, in the absence of any measurements.

[10 marks]

c) Using the Kalman filter equation, determine the expression for the best estimate of the state at time t_{i+1}, taking into account the measurement y_{i+1}.

[10 marks]

d) Using the Kalman filter equation, determine the expression for the uncertainty associated with the best estimate of the state at time t_{i+1}, taking into account the measurement y_{i+1}.

[10 marks]

e) Choose specific values for h and for the random process ε_{i+1} (i.e., h = 1 and r = 1) and use the values of relevant quantities chosen in b). Produce a sketch of the value of the best estimate of the state and its uncertainty at time t_2, when only one measurement $y_2 = 5$ at time t_2 is available.

[10 marks]

[End of Question paper]