Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.

Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.

April 2011

Answer Book
General Data Sheet
Any bilingual English language dictionary permitted
Only Casio-fx83 calculators are permitted

THE UNIVERSITY OF READING

MSc Examination for Courses in Sciences

Unit 2/MT/MD03: Monte-Carlo methods and Particle filters

MTMD03

2 hours

Answer ALL questions

The marks for the individual components of each question are given in [ ] brackets.
The total mark for the paper is 100.

1. (a) Question on sampling from simple distributions

Describe how one can use the transformation method, also called probability integral transform method, to derive samples from the Cauchy distribution given by

$$f(z) = \frac{1}{\pi} \frac{1}{1+z^2}$$

starting from samples from the uniform distribution U[0,1].
(Hint: transform z=tan y.)

[8 marks]

(b) Describe three methods to reduce the Monte-Carlo variance of a Monte-Carlo estimate of some function f(x).

[12 marks]

(c) In figure 1 a posterior pdf is given. Explain how you would implement the Gibbs sampler for this problem.

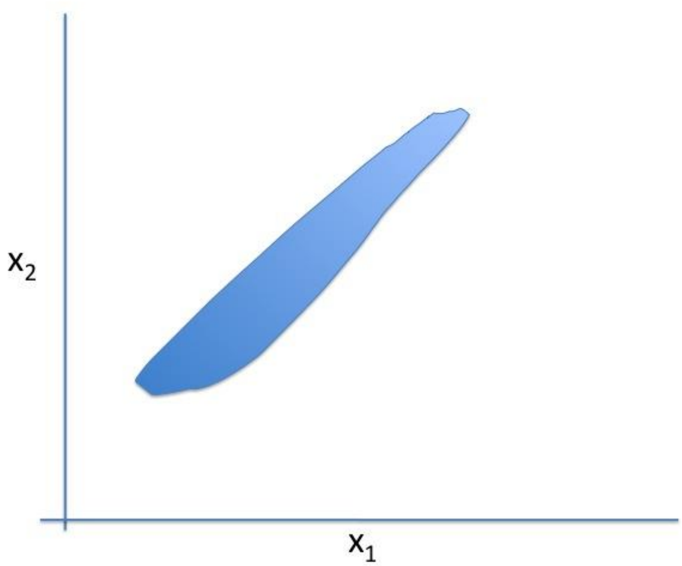

Figure 1

[8 marks]

Question 1 continued overleaf

Question 1 continued.

(c) Assume the target pdf is given by figure 2. Discuss if the Gibbs sampler can be used to explore this density. Do the same for the density displayed in figure 3.

[8 marks]

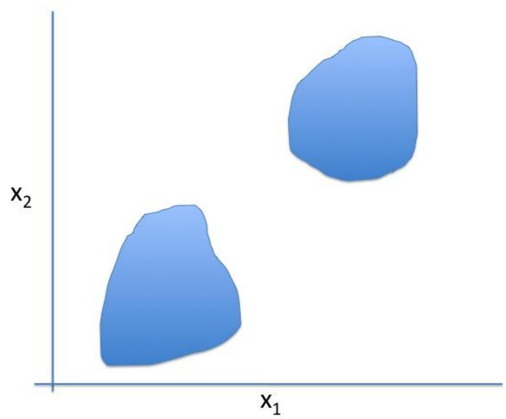

Figure 2

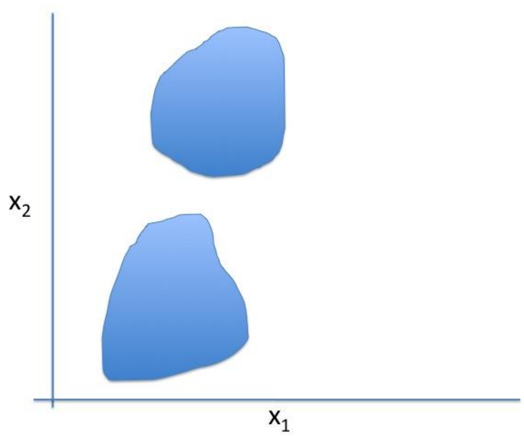

Figure 3

2.

(a) Consider a Markov chain in a 2-dimensional system with transition matrix

$$P = \begin{pmatrix} 1-p & r \\ s & 1-q \end{pmatrix}$$

Find conditions on p, q, r and s such that this is a proper transition matrix. Use these to eliminate r and s.
Find the stationary or equilibrium distribution π in terms of p and q.
Does detailed balance hold?
Is the stationary distribution also the limiting distribution for all values of p and q?

[10 marks]

(b) Show that performing the transition matrix m times gives:

$$P^m = \frac{1}{p+q}\begin{pmatrix} q & p \\ q & p \end{pmatrix} + \frac{\lambda^m}{p+q}\begin{pmatrix} p & -p \\ -q & q \end{pmatrix}$$

in which λ=1-p-q.
(Hint: Show this for m=1, and then use induction.)

[8 marks]

(c) Show that if we want

$$|\Pr(x^{(m)} = i \mid x^{(0)} = j) - \pi_i| \leq \varepsilon$$

for i,j = 0,1, so we want the transition to be within ε from its limiting value, then

$$m \geq \frac{\log\left(\dfrac{(p+q)\varepsilon}{\max(p,q)}\right)}{\log|\lambda|}$$

(Hint: Note that the first term in the expression for P^m is build up from the elements of the stationary distribution π, so concentrate on the second term.)
What does this tell you about the burn-in period?

[10 marks]

Question 2 continues overleaf

Turn over

Question 2 continued

(d) Explain how you would implement a Metropolis-Hastings algorithm using the knowledge gained in 2(c). Discuss ways to accelerate the convergence of the algorithm.

[8 marks]

Turn over

3.

(a) Explain why sequential importance sampling (the standard particle filter) needs a large number of particles when the number of independent observations is large.
How can resampling reduce the number of particles needed?
Explain one resampling method.

[10 marks]

(b) Explain how a proposal density can be used in Particle filtering to increase the efficiency of the particles.

[10 marks]

(c) In the Ensemble Kalman Filter localisation is used in large dimensional systems. Give two reasons for localisation.
Why does direct localisation lead to problems in a particle filter?
What would you do to reduce (or eliminate) the problem?

[8 marks]

End of question paper