Estimating Model Error Covariance (Q) using Implicit Equal-Weights Particle Filter

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Table Of Contents

The update equation for the IEWPF:

$$x_i^n = f(x_i^{n-1}) + \hat{K}(y - Hf(x_i^{n-1})) + \hat{P}^{1/2}\alpha_i^{1/2}\xi_i \qquad (1)$$

in which

$$\hat{K} = \hat{Q}H^T(H\hat{Q}H^T + R)^{-1}, \qquad (2)$$

and

$$\hat{P} = (\hat{Q}^{-1} + H^T R^{-1} H)^{-1} \qquad (3)$$

and $\hat{Q}$ is our current estimate of Q.

We know that:

1. Q is very important for IEWPF.
2. Q is very hard to model.

At first, we have that

$$y^n - Hf(x_i^{n-1}) = y^n - Hx_t^n + H(x_t^n - f(x_i^{n-1})) \tag{4}$$

If we expect that

$$x_a^n = \bar{x}^n \approx x_t^n \tag{5}$$

$$f(x_a^{n-1}) = f(\bar{x}^{n-1}) \approx f(x_t^{n-1}) \tag{6}$$

Using the tricks and assumptions, we can get that

$$\begin{aligned} y^n - Hf(x_i^{n-1}) &= y^n - Hx_t^n + H(x_t^n - f(x_t^{n-1}) + f(x_t^{n-1}) - f(x_i^{n-1})) \\ &= y^n - Hx_t^n + H(x_t^n - f(x_t^{n-1})) + H(f(x_t^{n-1}) - f(x_i^{n-1})) \\ &\approx \varepsilon_o^n + H\epsilon^n + H(f(\bar{x}^{n-1}) - f(x_i^{n-1})) \\ &= \varepsilon_o^n + H\epsilon^n + Hv_i^n \end{aligned} \tag{7}$$

So that, one way to estimate Q is to use

$$C = (\frac{1}{N_e - 1}) \sum_{i=1}^{N_e} (y - Hf(x_i^{n-1}))(y - Hf(x_i^{n-1}))^T \approx HQH^T + R + HVH^T \quad (8)$$

, which is under the assumption that the terms in the previous slide are uncorrelated. The estimate then becomes

$$HQH^T = C - R - HVH^T \quad (9)$$

Note that this method does not use the IEWPF update step and we know the exact R matrix.

We can also form:

$$C = \frac{1}{N_e - 1} \sum_{i=1}^{N_e} (y - Hx_i^n)(y - Hx_i^n)^T \tag{10}$$

Using the same tricks with the previous method, we can derive that

$$C = L_0 R L_0^T + L_0 HQH^T L_0^T + L_0 HVH^T L_0^T + W \tag{11}$$

in which we defined L_0 via:

$$L_0 = (1 - HK_0) \tag{12}$$

and

$$W = HP_0 \frac{1}{N_e - 1} \sum_{i=1}^{N_e} \alpha_i \xi_i {\xi_i}^T H^T \tag{13}$$

We thus find:

$$L_0 HQH^T L_0^T = C - L_0(R + HVH^T) L_0^T - W \tag{14}$$

Finally, one could estimate Q via:

$$C = \frac{1}{N_e - 1} \sum_{i=1}^{N_e} (y - Hf(x_i^{n-1}))(y - Hx_i^n)^T \tag{15}$$

Then, we can derive the model error covariance using the same tricks as:

$$HQH^T L_0^T = C - (R + HVH^T) L_0^T \tag{16}$$

Algorithm 1 Estimating Q

$Q0 \leftarrow s * B$
for $k \leftarrow 1, 200$ **do**
  $Q_e(k) \leftarrow Estimate\_Q(method = m1, m2, m3, Q = Q0)$
  $Q_m \leftarrow \frac{1}{k} \sum_{i=1}^{k} Q_e(i)$ {not used yet}
end for
for $k \leftarrow 201, 250$ **do**
  $Q_e(k) \leftarrow Estimate\_Q(method = m1, m2, m3, Q = Q_m)$
  $Q_m \leftarrow \frac{1}{k-1} \sum_{i=1}^{k-1} Q_e(i)$
  $Q_m \leftarrow (1 - \alpha) * Q_m + \alpha * Q_e(k)$ {use Q_m in IEWPF}
end for

Lorenz 96 Experiments

- 1,000 dimensional Lorenz 96 model
- 40 - 400 ensemble members.
- B is tridiagonal with values 1 and 0.25(only used to initiate algorithm).
- True Q is tridiagonal matrix with values 0.20 and 0.05.
- R is a diagonal matrix.
- All grid points are observed every time step.
- Localization with radius 5 grid points.

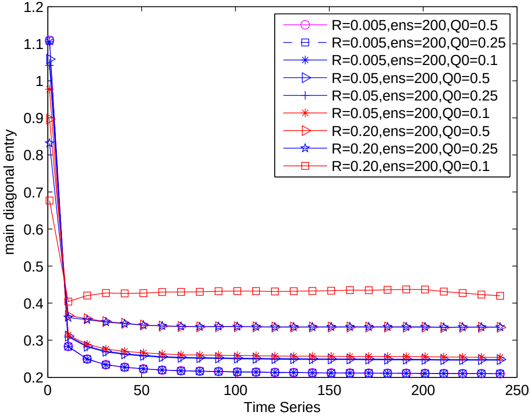

Figure: mean value of main diagonal entries(m1). The other methods are the same.

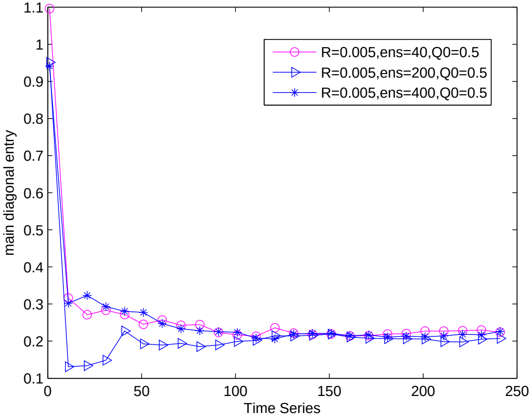

Figure: 500^{th} entry of main diagonal entries varies with ensemble size(m1).

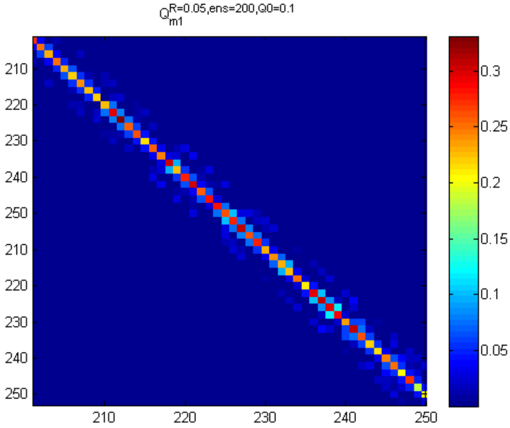

Figure: part of the Q matrix(m1). We should do smoothing along diagonals, and also the sub- and super- diagonal values.

Conclusions

- The implicit equal-weights particle filter has the potential to estimate Q without estimating B.
- The three methods are all iterative ways of estimating Q.
- The smaller diagonal values R have, the more accurate Q is.
- The initial first guess $Q0$ matrix influences the final results.
- The ensemble size has little impact on the final results.
- We have plenty work to do to improve the methods and to implement them on real models.

Thank you very much for attention.