Assimilation of Rayleigh-Bénard using only velocity data

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Data Assimilation by feedback nudging

[Azouani-Olson-Titi ‘13, Azouani-Titi ‘14]

$$\frac{du}{dt} = F(u) \qquad \text{know only } J_h u(t), \quad t_0 \le t$$

$$\frac{d\tilde{u}}{dt} = F(\tilde{u}) - \mu J_h(\tilde{u} - u) \qquad \text{IC arbitrary}$$

Under certain conditions on μ, J_h

$$\|\tilde{u}(t) - u(t)\| \to 0 \quad \text{as} \quad t \to \infty \quad \text{at exp rate}$$

2D Rayleigh-Bénard

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla) u + \nabla p = \vartheta \mathbf{e}_2 \\ \frac{\partial \vartheta}{\partial t} - \kappa \Delta \vartheta + (u \cdot \nabla) \vartheta - u \cdot \mathbf{e}_2 = 0 \\ \nabla \cdot u = 0 \end{cases}$$

$u, \vartheta = 0$ at $x_2 = 0$ and $x_2 = 1$,

u, ϑ, p are periodic, of period L, in the x_1-direction.

Data assimilation using velocity data only (Farhat-J.-Titi, *Phys D* 2015)

$$\begin{cases} \frac{\partial u}{\partial t} - \nu\Delta u + (u\cdot\nabla)u + \nabla p = \vartheta\mathbf{e}_2 \\ \frac{\partial \vartheta}{\partial t} - \kappa\Delta\vartheta + (u\cdot\nabla)\vartheta - u\cdot\mathbf{e}_2 = 0 \end{cases}$$

$$\begin{cases} \frac{\partial \tilde{u}}{\partial t} - \nu\Delta \tilde{u} + (\tilde{u}\cdot\nabla)\tilde{u} + \nabla \tilde{p} = \tilde{\vartheta}\mathbf{e}_2 - \mu J_h(\tilde{u}-u) \\ \frac{\partial \tilde{\vartheta}}{\partial t} - \kappa\Delta\tilde{\vartheta} + (\tilde{u}\cdot\nabla)\tilde{\vartheta} - \tilde{u}\cdot\mathbf{e}_2 = 0 \end{cases}$$

If $\mu > C$ and $\mu h^2 \lesssim \nu$ then as $t \to \infty$

$$\|u(t) - \tilde{u}(t)\|_{L^2} + \left\|\vartheta(t) - \tilde{\vartheta}(t)\right\|_{L^2} \to 0 \quad \text{at exp rate}$$

Simplest case, J_h =proj onto first $\lfloor 1/h \rfloor$ Fourier modes

Use $\|J_h(w) - w\|_{L^2} \le c_0 h \|w\|_{H^1}$ and $\mu c_0^2 h^2 \le \nu$ to extract dissipation

$$-\mu(J_h(w), w) = -\mu(J_h(w) - w, w) - \mu \|w\|_{L^2}^2$$

Cauchy-Schwarz

$$\le \mu \|J_h(w) - w\|_{L^2} \|w\|_{L^2} - \mu \|w\|_{L^2}^2$$

Young's

$$\le \frac{\mu c_0^2 h^2}{2} \|w\|_{H^1}^2 - \frac{\mu}{2} \|w\|_{L^2}^2$$

$$\le \frac{\nu}{2} \|w\|_{H^1}^2 - \frac{\mu}{2} \|w\|_{L^2}^2$$

Gronwall Estimate

Set $w = u - \tilde{u}$, $\xi = \vartheta - \tilde{\vartheta}$.
Take $(w \text{ eqn}, w) + (\xi \text{ eqn}, \xi)$
Use

$$-\mu(J_h(w), w) \leq \frac{\nu}{2} \|w\|_{H^1}^2 - \frac{\mu}{2} \|w\|_{L^2}^2$$

$$\vdots$$

$$\frac{d}{dt}\left(\|w\|_{L^2}^2 + \|\xi\|_{L^2}^2\right) + \lambda_1 \min\{\nu, \kappa\}\left(\|w\|_{L^2}^2 + \|\xi\|_{L^2}^2\right) \leq \underbrace{\left(\frac{4}{\kappa\lambda_1} + \frac{4c}{\nu}\|u\|_{H^1}^2 + \frac{4c}{\nu\kappa^2\lambda_1}\|\vartheta\|_{H^1}^4 - \mu\right)}_{\text{take } \mu \text{ big to make this } <0} \|w\|_{L^2}^2$$

$\|w\|_{L^2}^2 + \|\xi\|_{L^2}^2$ decays like $e^{-\lambda_1 \min\{\nu,\kappa\} t}$, $\quad \lambda_1 =$ smallest e-val

Resolution needed

$$\frac{4}{\kappa\lambda_1} + \frac{4c}{\nu}\,\|u\|_{H^1}^2 + \underbrace{\frac{4c}{\nu\kappa^2\lambda_1}\,\|\vartheta\|_{H^1}^4}_{\text{probably worst term}} < \mu \quad \text{and} \quad \mu h^2 \lesssim \nu$$

For $Pr = \dfrac{\nu}{\kappa} \sim 1$, this would mean in terms of $Ra = \dfrac{1}{\nu\kappa}$

$$\mu \gtrsim Ra^{3/2}\,\|\vartheta\|_{H^1}^4$$

[Foias-Manley-Temam 1987] On the global attractor

$$\|\vartheta\|_{H^1}^2 \le a e^b \quad a = \mathcal{O}(\nu^{-2}),\ b = \mathcal{O}(\nu^{-7})$$

$$\mu > Ra^{7/2} e^{Ra^{7/2}} \qquad \text{so} \qquad h \lesssim Ra^{-7/4} e^{-Ra^{7/2}}$$

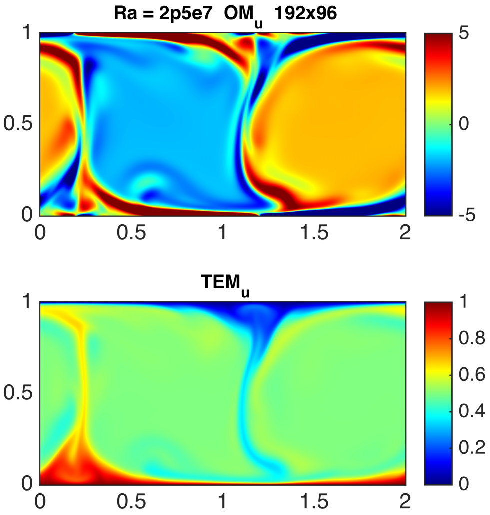
Computational results, w/ Hans Johnston, U.Mass

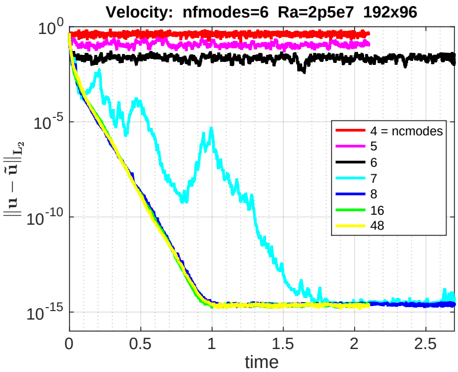
$$\omega = -\nabla \times u, \qquad u^\perp = \nabla\psi, \qquad \omega = \Delta\psi$$

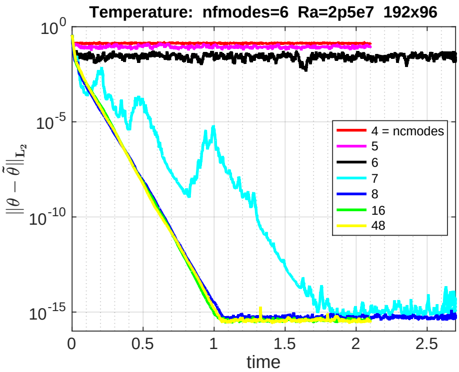
$$\frac{\partial \omega}{\partial t} - \nu\Delta\omega + (u \cdot \nabla)\omega = -\frac{\partial \theta}{\partial x_1}$$
$$\frac{\partial \theta}{\partial t} - \kappa\Delta\theta + (u \cdot \nabla)\theta = 0$$

$$\frac{\partial \tilde{\omega}}{\partial t} - \nu\Delta\tilde{\omega} + (\tilde{u} \cdot \nabla)\tilde{\omega} = -\frac{\partial \tilde{\theta}}{\partial x_1} - \mu J_h(\tilde{\omega} - \omega)$$
$$\frac{\partial \tilde{\theta}}{\partial t} - \kappa\Delta\tilde{\theta} + (\tilde{u} \cdot \nabla)\tilde{\theta} = 0$$

usual BC: temperature $= 1$ on bottom, 0 on top


Ra = 2p5e7 OM_u 192x96
1
0.5
0
0
0.5
1
1.5
2
5
0
-5
TEM_u
1
0.5
0
0
0.5
1
1.5
2
1
0.8
0.6
0.4
0.2
0


Velocity: nfmodes=6 Ra=2p5e7 192x96
$\|\mathbf{u}-\tilde{\mathbf{u}}\|_{\mathbf{L}_2}$
time
4 = ncmodes
5
6
7
8
16
48


Temperature: nfmodes=6 Ra=2p5e7 192x96
$\|\theta-\tilde{\theta}\|_{\mathbf{L}_2}$
time
4 = ncmodes
5
6
7
8
16
48

Another computational test on Rayleigh-Bénard

[Altaf-Titi-Knio-Zhao-McCabe-Hoteit 2015]

Low Rayleigh numbers

Finite volume elements $J_h(\varphi)(x) = \sum_{k=1}^{N_h} \varphi(x_k)\chi_{Q_k}(x)$

diam$(Q_k) < h$, $\quad \cup_{k=1}^{N_h} = \Omega = [0, 2] \times [0, 1]$

Assimilation with velocity alone is effective, but temp alone is not

Data assimilation in porous medium using **temperature** data only (Farhat-Lunasin-Titi 2015)

$$\begin{cases} \gamma \frac{\partial u}{\partial t} + u + \nabla p = Ra \vartheta \mathbf{e}_2 \\ \frac{\partial \vartheta}{\partial t} - \Delta \vartheta + (u \cdot \nabla)\vartheta - u \cdot \mathbf{e}_2 = 0 \end{cases} \qquad \nabla \cdot u = 0,$$

$$\begin{cases} \gamma \frac{\partial \tilde{u}}{\partial t} + \tilde{u} + \nabla \tilde{p} = Ra \tilde{\vartheta} \mathbf{e}_2 \\ \frac{\partial \tilde{\vartheta}}{\partial t} - \Delta \tilde{\vartheta} + (\tilde{u} \cdot \nabla)\tilde{\vartheta} - \tilde{u} \cdot \mathbf{e}_2 = -\mu J_h(\tilde{\vartheta} - \vartheta) \end{cases} \qquad \nabla \cdot \tilde{u} = 0$$

For μ sufficiently big, $\mu h^2 \lesssim \nu$

$$\|u(t) - \tilde{u}(t)\|_{L^2}^2 + \left\|\vartheta(t) - \tilde{\vartheta}(t)\right\|_{L^2}^2 \to 0$$

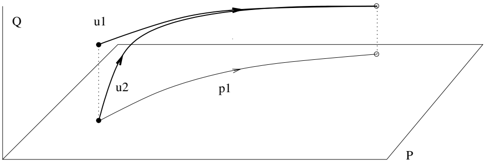
Still other works

- Azouani-Titi '13: 1D reaction-diffusion equation
- Azouani-Olson-Titi '13: 2D NSE
- Bessaih-Olson-Titi '14: 2D NSE, noisy data
- Farhat-Lunasin-Titi '14: 2D NSE, velocity data in one direction
- Albanez-Lopes-Titi '14: 3D NSE-α
- Markowich-Titi-Trabelsi '15: 3D Brinkman-Forchheimer-Darcy
- Foias-Mondaini-Titi '15: Statistics of data assim, 2D NSE
- J-Sadigov-Titi '15: Damped, driven KdV
- J.-Martinez-Titi '16: Subrcritical SQG
- Farhat-Lunasin-Titi '16: 3D planetary geostrophic, temp data only

Data Assimilation (classic) [Charney-Halem-Jastrow '69]

Given $p_1(\cdot)$, where $u_1(\cdot) = p_1(\cdot) + q_1(\cdot)$ for projs P, $Q = I - P$, solve

$$\frac{dq_2}{dt} + \nu A q_2 + QB(p_1 + q_2) = Qf, \quad q_2(t_0) = \eta \text{ (typically } = 0)$$



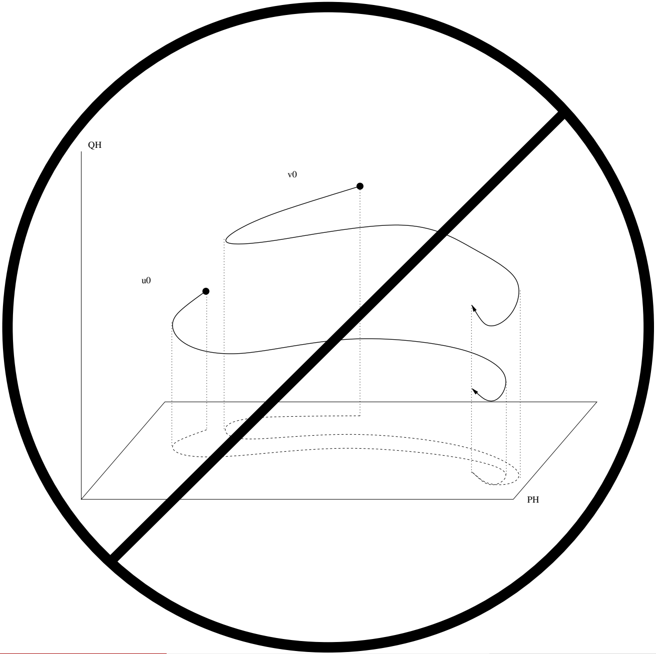
[Olson-Titi '03] cond on f s.t. $\|p_1(t) + q_2(t) - u_1(t)\| \to 0$ as $t \to \infty$

Determining modes

Foias-Prodi 1967
Foias-Manley-Temam-Tréve 1983
Constantin-Foias 1985
Constantin-Foias-Manley-Temam 1985
Hale-Raugel 2003

A projector P onto Fourier modes is *determining* if

$$\|Pv(t) - Pu(t)\| \to 0 \implies \|v - u\| \to 0 \quad \text{as} \quad t \to \infty$$


QH
v0
u0
PH

[Jones-Titi '93] For 2D NSE num det modes $\lesssim G = \frac{|f|}{\nu^2 \lambda_1}$
[Olson-Titi'08] Numerical evidence that num det modes much smaller

$$\frac{du_1}{dt} + \nu A u_1 + B(u_1) = f$$
$$\frac{du_2}{dt} + \nu A u_2 + B(Pu_1 + Qu_2) = f$$

waveno.s of modes in force $\in [10, 12]$
Given G, find min num modes in P such $|u_1 - u_2| \to 0$ as $t \to \infty$

Gr(f)	FFT	Δt	CFL_{max}	k_{max}	λ_c	n_c
12,500	512^2	0.16	0.2362	20.2	0	0
25,000	512^2	0.16	0.4188	28.9	8	24
37,500	512^2	0.16	0.7083	34.8	20	68
50,000	512^2	0.16	0.9467	40.8	26	88
62,500	512^2	0.08	0.2757	45.8	29	96
125,000	512^2	0.01	0.1217	60.0	34	108
250,000	512^2	0.01	0.2061	74.7	26	88
500,000	512^2	0.01	0.3328	104.4	25	80
750,000	512^2	0.01	0.4569	99.6	25	80
1,000,000	512^2	0.01	0.5453	113.6	25	80
1,250,000	512^2	0.005	0.3276	112.7	25	80
1,500,000	512^2	0.005	0.3892	123.0	25	80
1,750,000	512^2	0.005	0.4280	123.6	25	80
2,000,000	512^2	0.005	0.4380	137.4	25	80
2,500,000	512^2	0.005	0.5161	129.1	25	80
3,000,000	512^2	0.005	0.5663	148.8	25	80
4,000,000	512^2	0.005	0.7060	164.6	25	80
6,000,000	768^2	0.00125	0.3455	178.7	25	80
8,000,000	768^2	0.00125	0.4452	189.3	25	80
12,000,000	768^2	0.00125	0.5302	227.2	25	80
16,000,000	768^2	0.00125	0.7002	257.1	25	80
24,000,000	$1{,}024^2$	0.000625	0.5016	262.8	25	80
36,000,000	$1{,}024^2$	0.000625	0.7345	332.6	25	80
48,000,000	$1{,}024^2$	0.0003125	0.4303	344.2	25	80
60,000,000	$1{,}024^2$	0.0003125	0.4984	378.8	25	80