Toward variational data assimilation for coupled models: first experiments on a diffusion problem

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Context

- **Ocean-atmosphere coupled models** have a key role in weather forecast nowadays
- The coupling methods may severely impact the model solution. An exact solution of the coupling problem can be obtained using a **Global-in-Time Schwarz method** (Lemarié et al. [2014])
- The **initialisation of coupled models** also has a major impact on the forecast solution (Mulholland et al. [2015])
- Few **coupled DA methods** started to be developed (Smith et al. [2015], Laloyaux et al. [2015]...) for coupled systems, and showed promising results

Our approach

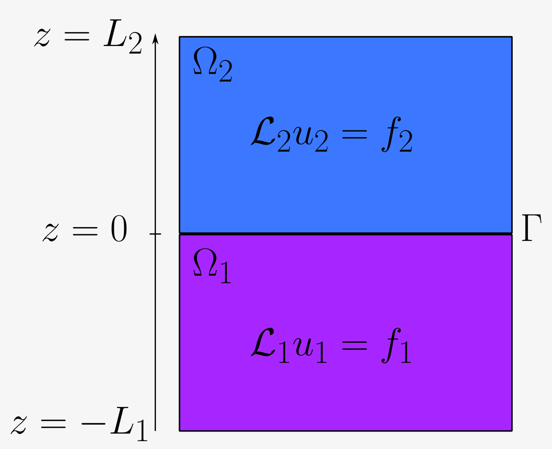
- The dynamical equations of our system are coupled using an **iterative Schwarz domain decomposition method** (Gander [2008])
- We are using **variational DA** techniques, which require minimization iterations and we are looking to **take benefit of the minimization iterations** to **converge toward the exact solution** of the coupling problem: the minimisations iterations substitute the Schwarz iterations
- Three general variational DA algorithms, are presented here and applied to a simple coupled system (Pellerej et al. [2016])

1. Model problem and coupling strategy

Let us define two models on each space-time domain $\Omega_d \times [0,T]$ $(d=1,2)$, with a common interface $\Gamma = \{z=0\}$.

Problem: How to **strongly couple** the two models at their interface Γ ?

$\Rightarrow$ We propose to use a global-in-time Schwarz algorithm (Gander [2008])



For a given initial condition $u_0 \in H^1(\Omega_1 \cup \Omega_2)$ and *first-guess* $u_1^0(0,t)$, the coupling algorithm reads

$$\begin{cases} \mathcal{L}_2 u_2^k = f_2 & \text{on } \Omega_2 \times T_W \\ u_2^k(z,0) = u_0(z) & z \in \Omega_2 \\ \mathcal{G}_2 u_2^k = \mathcal{G}_1 u_1^{k-1} & \text{on } \Gamma \times T_W \end{cases} \quad \begin{cases} \mathcal{L}_1 u_1^k = f_1 & \text{on } \Omega_1 \times T_W \\ u_1^k(z,0) = u_0(z) & z \in \Omega_1 \\ \mathcal{F}_1 u_1^k = \mathcal{F}_2 u_2^k & \text{on } \Gamma \times T_W \end{cases} \tag{1}$$

$\mathcal{F}_d$ and $\mathcal{G}_d$ are the interface operators, k is the iteration number, $T_W = [0,T]$, and $f_d \in L^2(0,T;L^2(\Omega_d))$ is a given right-hand side

- At convergence, this algorithm provides a mathematically strongly coupled solution which satisfies $\mathcal{F}_1 u_1 = \mathcal{F}_2 u_2$ and $\mathcal{G}_2 u_2 = \mathcal{G}_1 u_1$ on $\Gamma \times T_W$
- The convergence speed of the method greatly depends on the choice for $\mathcal{F}_d$ and $\mathcal{G}_d$ operators, and the choice of the *first-guess*

2. Classic data assimilation

Let us introduce the classic cost function for variational data assimilation in the **uncoupled case**, for a domain Ω_d

- $\mathbf{x}_{0,d} = u_{0,d}(z) = u_0(z),\ z \in \Omega_d$ $(d=1,2)$ is the controlled state vector

$$J_{Ucpl}(\mathbf{x}_{0,d}) = \overbrace{\left\langle \mathbf{x}_{0,d} - \mathbf{x}_d^b, \mathbf{B}^{-1}(\mathbf{x}_{0,d} - \mathbf{x}_d^b) \right\rangle_{\Omega_d}}^{J^b(\mathbf{x}_{0,d})} + \overbrace{\int_0^T \left\langle \mathbf{y} - H(\mathbf{x}_d), \mathbf{R}^{-1}(\mathbf{y} - H(\mathbf{x}_d)) \right\rangle_{\Omega_d} \mathrm{dt}}^{J^o(\mathbf{x}_{0,d})} \tag{2}$$

where $\langle \cdot \rangle_\Sigma$ is the usual Euclidian inner product on a spatial domain Σ.

3. Toward a coupled variational data assimilation

If the DA process is done separately on each subdomain, the initial condition $u_0 = (\mathbf{x}_{0,1}^a, \mathbf{x}_{0,2}^a)^T$ obtained on Ω does not satisfy the interface conditions. The interface imbalance in the initial condition can severely damage the forecast skills of coupled models (Mulholland et al. [2015])

Objective: properly take into account the coupling in the assimilation process

Full Iterative Method (FIM)

- $\mathbf{x_0} = u_0(z),\ z \in \Omega$
- We iterate the models till convergence of the Schwarz algorithm (k_{cvg} iterations)
- The first-guess u_1^0 in (1) is updated after each minimization iteration

$$J_{FIM}(\mathbf{x}_0) = J^b(\mathbf{x}_0) + \int_0^T \left\langle \mathbf{y} - H(\mathbf{x}^{\text{cvg}}), \mathbf{R}^{-1}(\mathbf{y} - H(\mathbf{x}^{\text{cvg}})) \right\rangle_\Omega \mathrm{dt} \tag{3}$$

where $\mathbf{x}^{\text{cvg}} = (u_1^{k_{\text{cvg}}}, u_2^{k_{\text{cvg}}})^T$

Truncated Iterative Method (TIM)

- $\mathbf{x}_0 = (u_0(z), u_1^0(0,t))^T$
- The Schwarz iterations are truncated at $k_{\max}$ iterations
- Extended cost function (misfit in the interface conditions) (Gejadze and Monnier [2007])

$$J_{TIM}(\mathbf{x}_0) = J^b(\mathbf{x}_0) + \int_0^T \left\langle \mathbf{y} - H\left(\mathbf{x}^{\text{trunc}}\right), \mathbf{R}^{-1}(\mathbf{y} - H\left(\mathbf{x}^{\text{trunc}}\right)) \right\rangle_\Omega \mathrm{dt} + J^s \tag{4}$$

where $J^s = \alpha_{\mathcal{F}} \|\mathcal{F}_1 u_1(0,t) - \mathcal{F}_2 u_2(0,t)\|^2_{[0,T]} + \alpha_{\mathcal{G}} \|\mathcal{G}_1 u_1(0,t) - \mathcal{G}_2 u_2(0,t)\|^2_{[0,T]}$ with $\|a\|^2_\Sigma = \langle a, a\rangle_\Sigma$ and $\mathbf{x}^{\text{trunc}} = (u_1^{k_{\max}}, u_2^{k_{\max}})^T$

Coupled Assimilation Method with Uncoupled models (CAMU)

- $\mathbf{x}_0 = (\mathbf{x}_{0,1}, \mathbf{x}_{0,2})^T$ with $\mathbf{x}_{0,d} = (u_0|_{z \in \Omega_d}, u_d^0(0,t))$
- We suppress the coupling between both models

The cost function for the CAMU is

$$J_{CAMU}(\mathbf{x}_0) = \left\{ \sum_{d=1}^{2} (J^b(\mathbf{x}_{0,d}) + J^o(\mathbf{x}_{0,d})) \right\} + J^s \tag{5}$$

Algo	Control vector	# of coupling iterations	extended cost function	Adjoint of the coupling	Coupling
FIM	$(u_0(z))$	k_{cvg}	no	yes	strong
TIM	$(u_0(z), u_1^0)^T$	$k_{\max}$	yes	yes	~strong
CAMU	$(u_0(z), u_1^0, u_2^0)^T$	0	yes	no	weak

Table 1 : Overview of the properties of the coupled variational DA methods described

The originality of these algorithms is the use of a **Schwarz algorithm** to couple our models jointly to the DA process with an **extended cost function**.

4. Application to a 1D diffusion problem

Previous algorithms are applied on a **1D linear diffusion problem**. We consider:

- $\mathcal{L}_d = \partial_t + \nu_d \partial_z^2$
- $\nu_1 \neq \nu_2$ the diffusion coefficients in each subdomain
- $\mathcal{F}_d = \nu_d \partial_z$ and $\mathcal{G}_d = \mathrm{Id}$ the interface operators on Γ (Dirichlet-Neumann)
- $u_d^\star(z,t) = \frac{U_0}{4} e^{-\frac{|z|}{\alpha_d}} \left\{3 + \cos^2\left(\frac{3\pi t}{\tau}\right)\right\}$ on $\Omega_d \times T_W$ the analytical solution

Single column observation experiment:

- Observations are available in $\Omega \setminus \{\Gamma\}$ at the end of the time-window (i.e. at $t = T$)
- We define the **interface imbalance indicator**, equal to J^s with $\alpha_{\mathcal{G}} = 0.01$ and $\alpha_{\mathcal{F}} = 40$

5. Single column observation experiment results

Algo	$\alpha_{\mathcal{G}}$	$\alpha_{\mathcal{F}}$	$k_{\max}$	# of minimisation iterations	# of models runs	Interface imbalance indicator	RMSE in °C
FIM	-	-	k_{cvg}	58	1169	$3.69\ 10^{-12}$	0.220
TIM	0	-	k_{cvg}	48	2016	$5.63\ 10^{-12}$	0.220
TIM	0	-	5	245	1225	$2.91\ 10^{-2}$	0.216
TIM	0	-	2	1518	3036	3.77	0.272
TIM	0.01	-	2	425	850	$9.89\ 10^{-7}$	0.217
TIM	**0.01**	-	**1**	**344**	**344**	$\mathbf{8.38\ 10^{-7}}$	**0.215**
CAMU	0.01	40	0	2957	2957	$1.40\ 10^{-4}$	0.231
CAMU	**0.001**	**4**	**0**	**268**	**268**	$\mathbf{9.38\ 10^{-3}}$	**0.240**
CAMU	0.0001	0.4	0	742	742	$3.29\ 10^{-1}$	0.327
Uncoupled	0	0	0	101	101	29.0	1.717

Table 2 : Results obtained for the three coupled variational DA methods

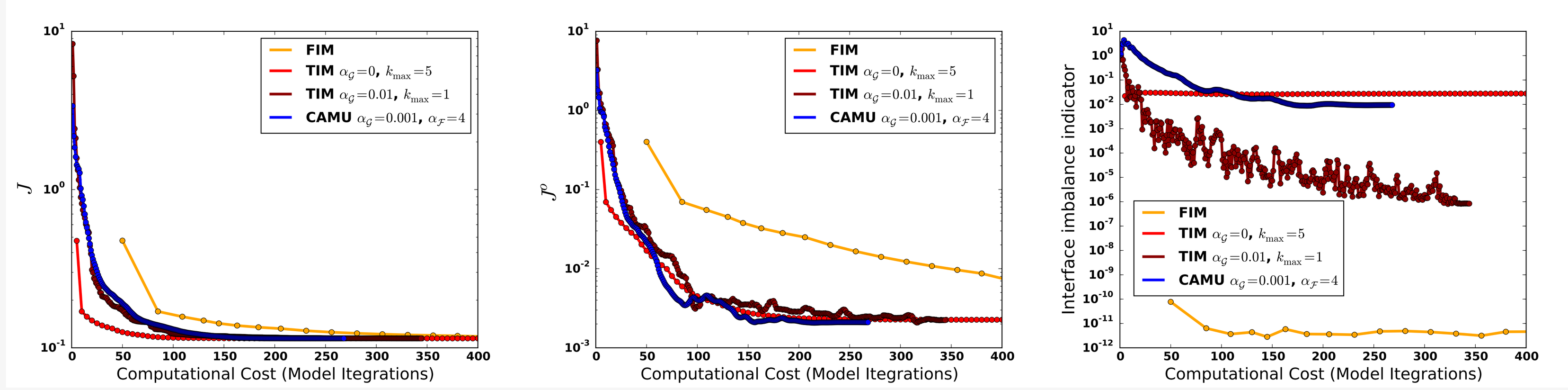


Figure 1 : Evolution of different terms with respect to number of model iterations for few configurations

6. Conclusions and perspectives

In the framework of an **iterative coupling**, we set up few data assimilation algorithms.

- Adding a physical constraint on the interface conditions in the cost function can have a beneficial effect on the performance of the method and allow to save coupling iterations
- An approach which only requires the adjoint of each individual model but not the adjoint of the coupling showed promising results
- The methods are very sensitive to the parameters choices
- We only test the algorithms on a simple linear problem

Perspectives

- Algorithm **convergence** and **conditioning** problem when J_S is part of the cost function will be studied
- Since the objective is to apply such methods to ocean-atmosphere coupled models, increasingly complex models including **physical parameterisations** for subgrid scales, and **non-linearities** will be considered

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