Influence of observations with correlated errors

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Introduction

In an idealised framework we study the influence of observations on the analysis as a function of correlation length scale.

It can be shown that for any correlation matrix **C** that its determinant is

- bounded above by 1 when **C** is equal to the identity and
- tends to zero as the correlation length scales of the matrix are increased.

Therefore, it can be argued that observations with correlated errors have less uncertainty than if they were uncorrelated and so we can expect them to be more informative.

However, the influence of the observations on the analysis depends not only on their uncertainty but also on the uncertainty in the a-priori information and the mapping between the observations and the state.

Measures of influence

The Sensitivity matrix:

Quantifies the sensitivity of the analysis in observation space, $\mathcal{H}(\mathbf{x}_a)$, to the observations, y.

$$\mathbf{S} = \frac{\partial \mathcal{H}(\mathbf{x}_a)}{\partial \mathbf{y}} \approx \mathbf{HK}$$

where **H** is the linearised observation matrix and **K** is the Kalman Gain (Cardinali Et al. 2004).

Degrees of freedom for signal:

Quantifies the degrees of freedom of a measurement related to signal, and is given by the expected value of the background cost function evaluated at the analysis (Rodgers, 2000).

$$d_s = E[J_b(\mathbf{x}^a)] = E\left[(\mathbf{x}^a - \mathbf{x}^b)\mathbf{B}^{-1}(\mathbf{x}^a - \mathbf{x}^b)\right]$$

$$d_s = trace(\mathbf{S}) = \sum_{k=0}^{p-1} \lambda_k,$$

where λ_k is the kth eigenvalue of the sensitivity matrix.

Mutual information:

Quantifies the reduction in uncertainty after an observation is made, and is therefore a function of the background error covariance, **B**, and the analysis error covariance, $\mathbf{P}_a$ (also referred to as Shannon Information content, Rodgers, 2000).

$$MI = 0.5 \ln |\mathbf{B}\mathbf{P}_a^{-1}|.$$

$$MI = -0.5 \ln |\mathbf{I}_p - \mathbf{S}| = -0.5 \sum_{k=0}^{n-1} \ln(1-\lambda_k)$$

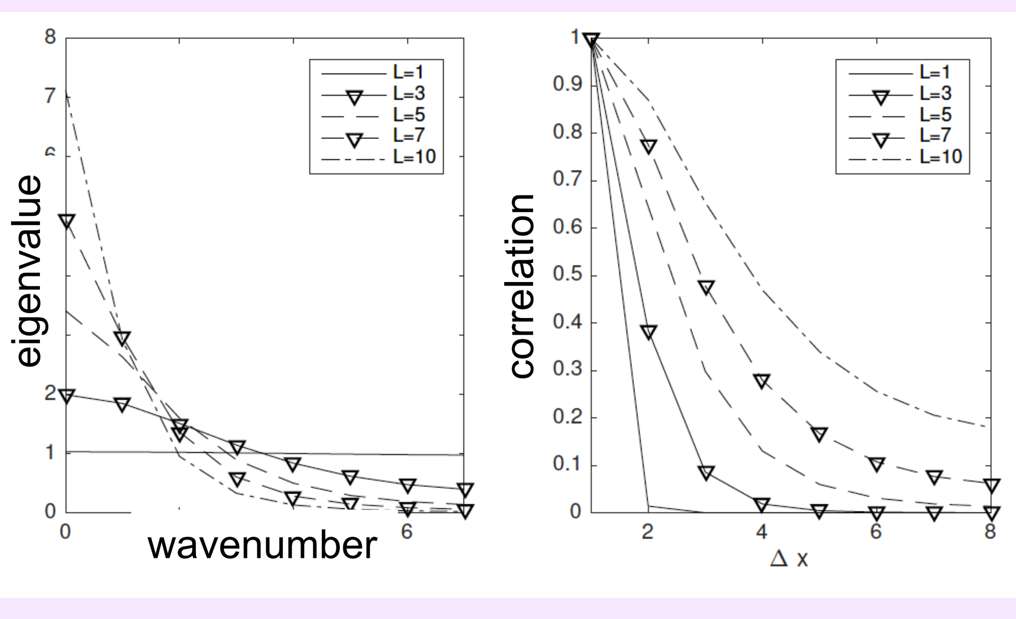

Figure 1. The eigenvalues of the SOAR function as a function of wavenumber (left). The correlations as a function of grid point (right)

Idealised framework

We consider the case when the observation error covariance matrix, $\mathbf{R}$, and the background error covariance matrix mapped to observation space, $\mathbf{HBH}^\mathrm{T}$ can both be described by circulant matrices. In this case they can be shown to have identical eigenvectors, $\mathbf{F}$ (Gray, 2006):

$$\mathbf{HBH}^\mathrm{T} = \beta \mathbf{F}\mathbf{\Gamma}\mathbf{F}^\mathrm{T}$$
$$\mathbf{R} = \rho \mathbf{F}\mathbf{\Psi}\mathbf{F}^\mathrm{T},$$

The eigenvalues are ordered by wavenumber, with the first relating to the eigenvector with the largest length scales.

In this case the sensitivity matrix is also a circulant matrix:

$$\mathbf{S} = \mathbf{HBH}^\mathrm{T}(\mathbf{HBH}^\mathrm{T} + \mathbf{R})^{-1} = \beta \mathbf{F}\mathbf{\Gamma}(\beta\mathbf{\Gamma} + \rho\mathbf{\Psi})^{-1}\mathbf{F}^\mathrm{T}$$

and hence its eigenvalues are given by

$$\lambda_k = \frac{\beta\gamma_k}{\beta\gamma_k + \rho\psi_k}$$

The SOAR correlation function:

In the following results it is assumed that **H=I** and the correlations of **B** and **R** can both be described by a SOAR function:

$$c_k = (1 + r_k/L)e^{-r_k/L}.$$

where r_k is the separation between two points and L is the correlation length scale. Here we assume a circular domain split into 16 equally space points. An example of the correlation function and its corresponding eigen spectrum is given in figure 1 for different values of L.

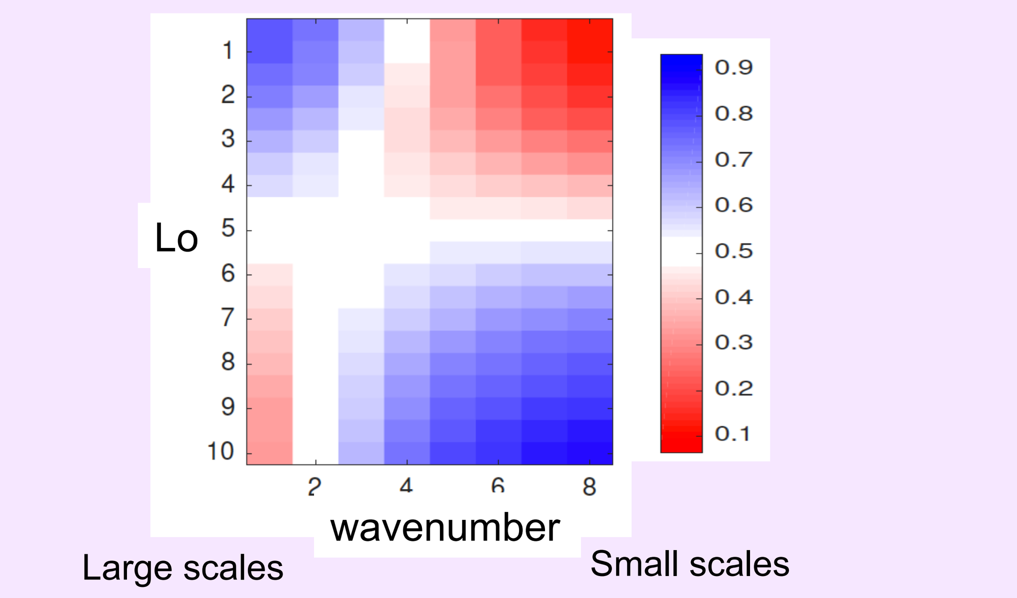

Figure 2. The eigenvalues of the sensitivity matrix as a function of wavenumber and Lo when Lb=5, beta=1, rho=1.

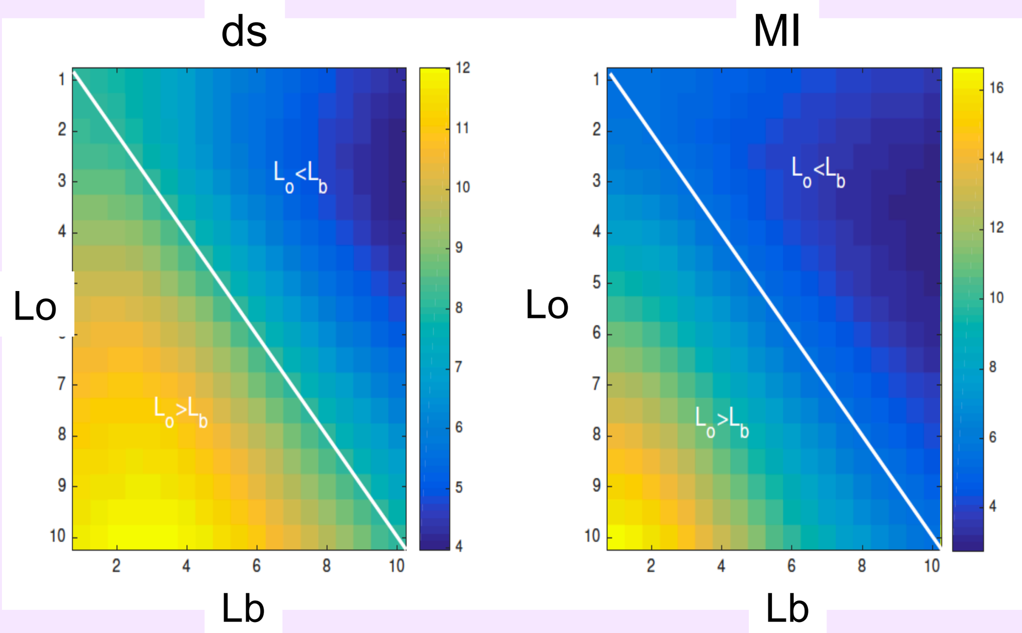

Figure 3. ds (left and MI (right) as a function of Lo and Lb when beta=1, rho=1.

References

1. Cardinali et al. 2004: Influence-matrix diagnostic of a data assimilation system. QJRMetSoc. 130, 2767–2786.
2. Rodgers 2000: Inverse methods for atmospheric sounding. World Scientific.
3. Gray. Toeplitz and Ciculant matrices: A Review. Now Publishers, 2006.

Results

L_o is the correlation length scales of the observation errors. L_b is the correlation length scales of background errors

- As L_o increases we see that the analysis becomes less sensitive to the observations at large scales and more sensitive to observations at small scales (Figure 2). However, if the analysis is more or less sensitive to observations at the small scales compared to the large scales depends on the ratio of L_o to L_b.
- In general, as L_o increases the overall information content of the observations increases. However, if L_b is large then increasing L_o may lead to a decrease in the total information content (Figure 3).
- The sum of the analysis error variances are at a maximum when L_o and L_b are similar (Figure 4.)

$$trace(\mathbf{P}_a) \overset{\mathbf{H=I}}{=} \sum_{k=0}^{p-1} \frac{\rho\beta\psi_k\gamma_k}{\beta\gamma_k + \rho\psi_k}$$

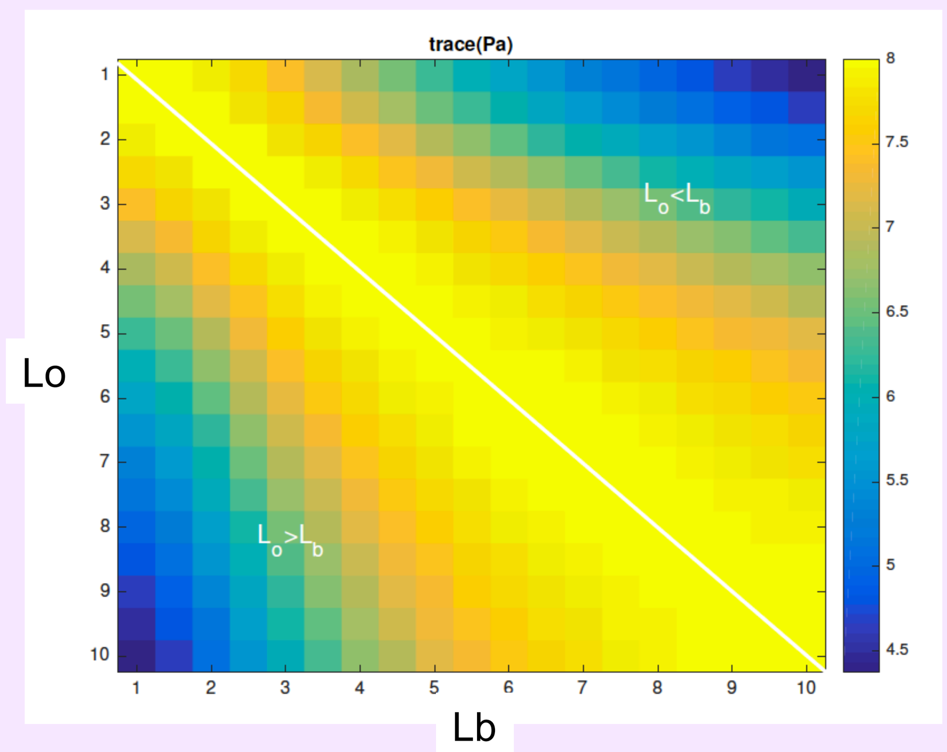

Figure 4. The trace of the analysis error covariance matrix as a function of Lb and Lo when beta=1, rho=1.

Conclusions

With the increasing efforts to explicitly account for observation error correlations in data assimilation it is important to have an understanding of how this changes the influence of the observations.

- The interactions between the correlations in the background and observation errors are very important for predicting the influence of the observations on the analysis.
- Observations with large error correlation lengthscales do not necessarily have more information than observation with uncorrelated errors if the background error correlation lengthscales are large.
- The greatest analysis accuracy is achieved when L_o and L_b are not equal. If L_o=L_b then in this idealised case $\mathbf{P}_a$ is proportional to **B**.

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