School of Mathematical and Physical Sciences

Accounting for Model Error: Weak Constraint 4DVar Assimilation

N.K. Nichols | A. El-Said | A.S. Lawless

Introduction

The aim of weak constraint 4-dimensional variational data assimilation (wc4DVAR) is to provide a statistically optimal estimate of the state of a dynamical system given an imperfect model of the system, a model forecast, and observations of the states over time. It differs from conventional 4DVAR by providing estimates for the model errors as well as the states.

We consider two different formulations of the problem and examine the behaviour of these as a function of

- **observation accuracy and configuration**
- **forecast accuracy and error covariance structure**
- **model error variance and correlation length scales**
- **assimilation window length .**

Weak-Constraint 4DVAR Formulations

[1] Finds the **initial state** x_0 and **model errors** $\{\eta_i\}$ by minimizing:

$$\min_{(x_0,\eta_i)} \mathcal{J}(x_0,\eta_i) = ||x_0 - x_0^b||^2_{B_0^{-1}} + \sum_{i=1}^{n} ||\mathcal{H}_i(x_i) - y_i||^2_{R_i^{-1}} + ||\eta_i||^2_{Q_i^{-1}}$$

subject to the *weak constraint*: $x_i = \mathcal{M}_i(x_{i-1}) + \eta_i$

where $\eta_i \sim N(0, Q_i)$

[2] Finds **all of the states** $\{x_i\}$ over the window by minimizing:

$$\min_{(x_i)} \mathcal{J}(x_i) = ||x_0 - x_0^b||^2_{B_0^{-1}} + \sum_{i=1}^{n} ||\mathcal{H}_i(x_i) - y_i||^2_{R_i^{-1}} + ||x_i - \mathcal{M}_i(x_{i-1})||^2_{Q_i^{-1}}$$

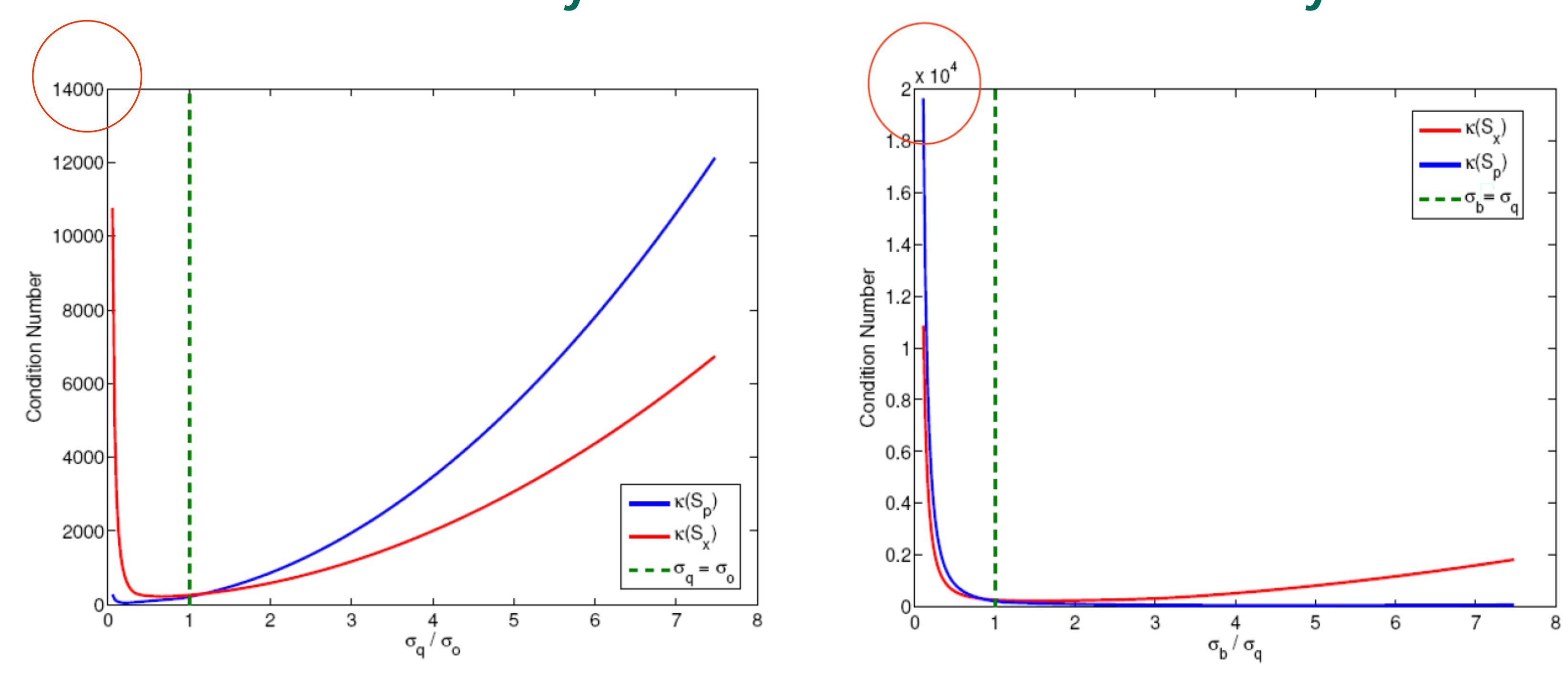
Comparison

The optimization problems [1] and [2] are solved by a gradient iteration procedure. The **condition number** of the Hessian of the objective function provides an estimate of the **accuracy** and **rate of convergence** of the solver. The Hessians have different structures and although the optimization problems are mathematically equivalent, the conditioning of the two problems is different and the numerical solvers have different convergence behaviour.

We investigated the behaviour of the condition number and convergence of the two formulations via **numerical examples**. Results are shown here using a linear **advection model** with periodic boundary conditions. For general systems, theoretical bounds on the conditioning of the two formulations have been derived that support the computational results[1] .

Results

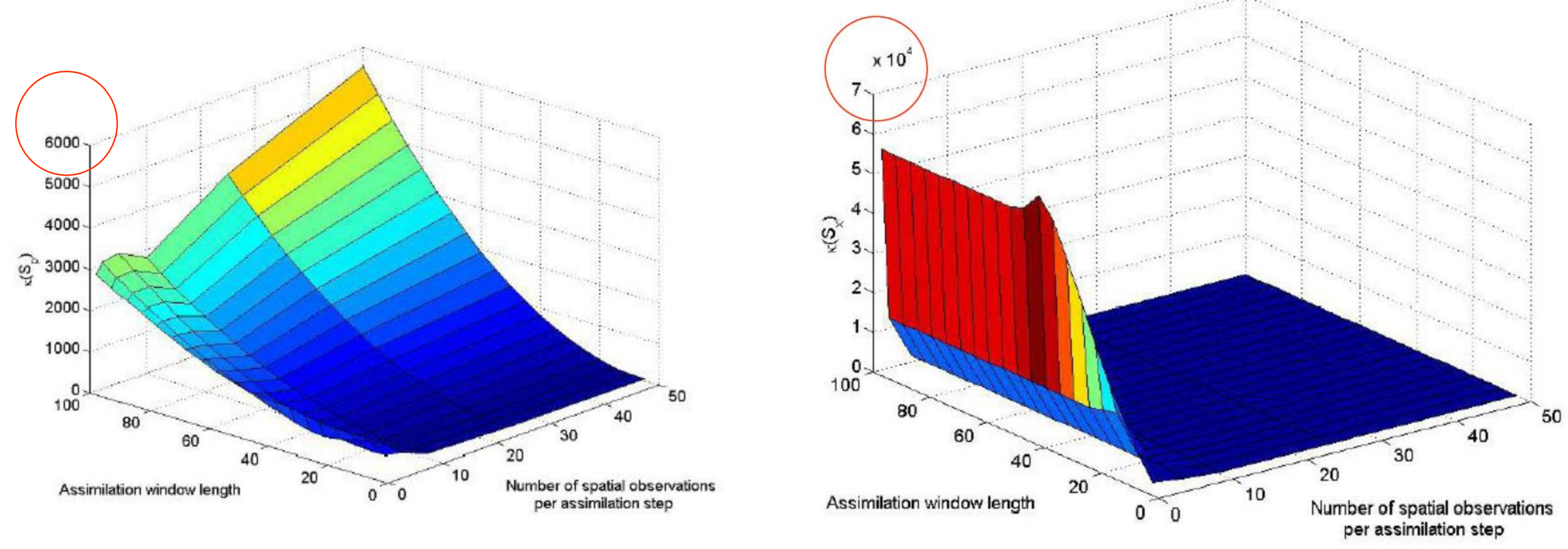
Observation accuracy / Forecast accuracy



Caption: Condition number of the Hessians of the formulations [1] (blue) and [2] (red) as a function of the model error variance relative to the observation error variance (left), and as a function of the forecast error variance relative to the model error variance (right).

We see that the **conditioning increases as** the **observation accuracy increases** (ie the observation error variance decreases) and also **increases as** the **model error variance increases** relative to the observation error and relative to the forecast error.

Window length



Caption: Condition number of the Hessians of [1] (left) and [2] (right) as a function of the assimilation window length and the number of observations in the window.

We see that the **conditioning of [1] increases** as the assimilation window length increases and as the observations become denser. The **conditioning of [2]**, however, **does not grow with the window length**, provided sufficient observations are available.

Conclusions

- The two different formulations of wc4DVar have different structures and are sensitive to the input data in different ways.
- The conditioning of the problem, and hence the work needed to solve the problem, increases as the observations become more accurate and as the model error becomes large.
- The conditioning of formulation [2] remains low for long windows provided enough observations are available.
- Formulation [1] can be preconditioned as for strong constraint 4DVar and the conditioning improved significantly, but preconditioning of [2] is an **open problem**.

References

1. A. El-Said, Conditioning of the weak-constraint variational data assimilation problem, for numerical weather prediction, *University of Reading, Department of Mathematics and Statistics, PhD thesis, 2015.* http://www.reading.ac.uk/web/FILES/maths/ElSaidThesis.pdf

Contact information

- Department of Mathematics and Statistics, University of Reading, Whiteknights, RG6 6AX
- **Email:** Nancy Nichols – n.k.nichols@reading.ac.uk ; Adam El-Said - adam.el-said@open.ac.uk ; Amos Lawless - a.s.lawless@reading.ac.uk .